Gyermeknevelés Tudományos Folyóirat
12. évf., 2. szám 5–23. (2024)
https://doi.org/10.31074/gyntf.2024.2.5.23

Plane-sphere comparative geometry: An experiment in the third grade of primary school

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Abstract
The main aim of our research was to teach basic concepts of spherical geometry to elementary school children, constantly comparing the spherical concepts with those of plane geometry. Twenty-eight third-grade students participated in playful activities dealing with elements of plane geometry and spherical geometry simultaneously. We have found that it becomes completely natural for third graders to compare two different worlds of geometry. This activity is beneficial not only for introducing spherical geometry, but also for a deeper understanding of planar geometry. In addition, spherical geometry contributes to better understanding of geographical concepts and orientation on the earth-globe. The post-test results confirmed our assumption about the advantages of comparative plane-sphere geometry in lower grades. Children who were considered less gifted in the subject showed interest and activity in these classes. Our experience suggests that further research on this topic may be necessary and fruitful with a larger sample of students.

Keywords: mathematics, spherical geometry, primary school

Introduction

In his book *The Number Sense*, Stanislas Dehaene (2011) states that spatial thinking and mathematics are “almost as if they were one and the same skill” (p. 135). Clements and Sarama (2011) argue that we convey ideas through mathematics that are fundamentally spatial in nature. Even something as simple as comparing shapes or numbers becomes spatial thinking when they are positioned differently on a number line, plane, or space. Interestingly, research suggests that the use of spatial representations becomes even more important as we progress in learning mathematics (Mix & Cheng, 2012).

The close relationship between spatial thinking and mathematics raises the possibility that the development of children's spatial abilities can be a key element in increasing the effectiveness of mathematics learning. Research is increasingly agreeing that spatial thinking plays a fundamental role in the early development of math skills. Farmer et al. (2013) found evidence that 3-year-old children's spatial skills are strong predictors of these same children's math performance by the time they start school. In another study, Verdine et al. (2014) reached a similar conclusion. It is worth noting that in both of the above studies the researchers used a relatively simple tool to assess children's spatial abilities. Children were shown shapes and bodies made of building blocks and were asked to copy them as accurately as possible. In their longitudinal study, Wolfgang et al. (2001) followed children of preschool age until adulthood. The entire duration of the research was 16 years. They showed that the complexity of building with building blocks at age 5 was a significant predictor of later high school math performance.

Researches argue that elementary geometry should be "the study of objects, motions, and relationships in a spatial environment" (Battista & Clements, 1988, p. 11). This means that the students' first geometric experiences are the informal study of concrete, tangible shapes and the properties of these shapes. The primary goal of the lessons should be to develop students' intuition and knowledge of the spatial environment. From the research of recent decades (e.g. Tzuriel & Egozi, 2010), we can conclude that activities related to drawing can also be effective in developing young children's spatial thinking. At the same time, the representations used during spatial thinking are internal thoughts, each student's own, so they are often difficult to show, since it is difficult to externalize and share them (Whiteley & Mamolo, 2014). That is why it is worthwhile to broaden the spatial geometry vocabulary of the students in the lessons, and to motivate them explaining to each other the thoughts that arise during the spatial geometry exercises and activities (Hawes et al., 2015).

> A crucial element of mathematical awareness lies in the formulation of the mathematical concept into words. At the beginning it is beneficial for the children to use self-coined expressions to name a shape. In the introductory phase, mathematical awareness is equally well served by a loose or »childish« word if it was invented by the child itself (C. Neményi, 2007, p. 23).

The starting point of our research was an educational experiment carried out in two classes in two Italian schools (Gambini, 2021). Fifth grade and older students studied spherical and plane geometry alongside for five school years. The aim was to provide a clearer understanding of geometric concepts and to give students satisfaction and self-confidence through the geometric experiments they carried out through the activities with the Lénárt sphere. Results have shown that simultaneous activities in plane geometry and spherical geometry help understand the properties of geometric shapes. They increase students' awareness of plane geometry throughout the school years, and support students' interest and progress in other areas of mathematics.

This experiment inspired us to introduce playful activities and tasks in this topic for younger children. Importantly, too early abstraction without sufficient empirical basis leads to a serious defect in the development of symbolic thinking, with long-term negative consequences for students' performance in mathematics.

We have always had in mind the implementation of spherical geometry through illustrations and activities. According to Piaget (Piaget & Inhelder, 2004), visual representation of geometric shapes begins at the age of 8–9 years, and the perception of volume appears only at the age of 11–12 years. Bruner's representation theory (Pintér, 2013) suggests that teaching spatial geometry is most effective at the enactive level, using tools, manipulations and activities. We used oranges, globes and Lénárt spheres to study concepts of spherical geometry. The earth-globe made it possible to combine geometry and geography. Children often encounter the same concept in different classes without noticing the equivalence. In lower grades, where one and the same teacher deals with different subjects, there is a special opportunity to develop spatial thinking in different topics, such as geometry and geography. Two- and three-dimensional orientation can be developed by reading maps and observing the globe (Chrappán, 2009).

Environmental studies for the third and fourth grades of the 2020 National Core Curriculum, and the framework curriculum for the year 2018 suggest studying spatial orientation, directions, landscapes, not only using maps, but also the globe (Environmental knowledge framework, 2020). Therefore, it was also possible to include the globe in spherical geometry lessons using the Lénárt sphere.

Methods

Twelve girls and sixteen boys from the 3rd grade of the Fazekas Mihály primary and secondary school in Budapest took part in the experiment during seven consecutive mathematics lessons. Most of the children had a positive attitude towards mathematics, which was largely due to their excellent teacher, Csilla Farkasházi. The children had only had a few geometry classes before the experiment, so many of the concepts were absolutely new to them.

In the spherical geometry classes, we dealt with the sphere and other solids in three-dimensional Euclidean space. We compared geometric shapes on the two-dimensional spherical surface with the corresponding shapes on a plane, such as the straight line, circle, polygons, etc. We performed geometric experiments and measurements on a sphere and an earth-globe. We measured the circumference of a spherical straight line or great circle on an orange and the distance of cities on the globe to interpret the concept of length and distance on different surfaces. (For the design and implementation of the spherical geometry lessons we used István Lénárt's textbook (Lénárt, 2009), his teaching aids (Teaching online modules) and his ideas.)

The first lesson began with freehand drawing on the surface of an orange. We use oranges for drawing because they are almost spherical, fit well in the hands of small children, and the peel can be easily drawn with a felt-tip pen. Children were asked to draw freely on the orange to make maximum use of the entire spherical surface. Then connect the drawings with the shortest possible path between them. We then discussed what a straight line looks like on a flat piece of paper and on a spherical orange. (Figure 1)

Figure 1

Finding the shortest route

We started the second lesson with a different visualization of a spherical line. The children continued working with the oranges, they were given colorful rubber rings and they were to use them to show spherical lines on the orange. (Figure 2)

Figure 2

Visualization of spherical lines with rubber rings

Then the children were asked to put a dot on the orange. We called it the North Pole. We told them that a penguin wants to be as far away from this North Pole as possible. Locate this farthest spot for the penguin on the orange. Another character, a turtle really hates cold weather and wants to be as far away from the frozen North and South Poles as possible. We asked the children to locate the turtle on the orange. (Figure 3)

Figure 3
Where does the turtle live?

In the third lesson, we repeated the concept of a spherical line and opposite points with the children. We asked them how many ways they could connect opposite points, and the children answered correctly in multiple ways. We also looked at a cantaloupe, which had spherical lines very clearly visible. The children noticed it at once. (Figure 4)

Figure 4
Spherical lines on the cantaloupe

Apples and orange peels were used to show the spherical biangles which were apparent on the orange peel. We asked the children about the properties of the biangle: its shape, the number of vertices and sides. They correctly answered that the biangle has two vertices and two sides. They were asked to try to draw a shape with two sides and two vertices on the paper. Most children drew shapes whose sides were not straight. (Figure 5)

Figure 5

Spherical biangle on paper

It was not easy to find the planar equivalent of the two sides of the biangle on the orange peel, to visualize the relationship between the spherical configuration and the planar representation. We also used apples to demonstrate spherical triangles. We talked about how a spherical triangle has three vertices and three sides. We finished this lesson with a game in which the children had to draw different (two, three or four) symbols on paper balls divided into eight equal parts, so that the same symbol could not be placed in adjacent parts. (Figure 6)

Figure 6

Symbols on the paper ball

In the fourth lesson, during the teacher's presentation, we asked the children to draw two spherical perpendicular straight lines on a paper ball. Then we asked them to write numbers 1, 2, 3, 4 in the regions, so that 1 meant spring, 2 meant summer, 3 meant autumn, and 4 meant winter (Figure 7). We asked them to rotate the paper ball and show how the seasons follow each other. We then asked them to tell how many seasons passed from spring to winter in the same year by rotating a paper ball; how many seasons pass between this summer and next year's winter; how many seasons are there between last autumn and next year's autumn? The children did very well using the paper ball to answer the questions.

Figure 7

Visualising seasons on the paper ball

It was a very encouraging experience that the paper ball (sphere) was also suitable for presenting an activity related to the divisibility of numbers.

Next, with the teacher's help, they drew another spherical line perpendicular to both perpendiculars. They got eight identical triangles on the entire surface. This network was used to introduce a spherical number puzzle which reminded the children of the well known Sudoku puzzle. They entered numbers 1, 2, 3, 4 in the four triangles respectively on a hemisphere. Then we discussed that additional numbers should be placed in the empty parts in such a way that the sum of the numbers in one hemisphere must always be 10. We asked them to write number 5 number 1, and to fill in the remaining three places according to the given rules. (Figure 8)

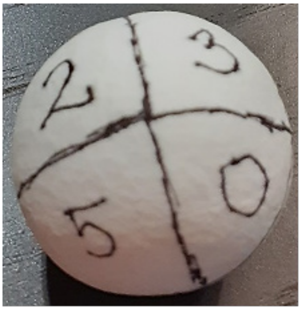
Figure 8

Number game on the paper ball 1

Then numbers 1, 2, 3, 4 were written again on a hemisphere, but number 5 was placed under 2, and the children filled in the three blank triangles. (Figure 9) In both tasks, we constantly checked the solutions. The children enjoyed the puzzle very much and correctly justified every answer.

Figure 9

Number game on the paper ball 2



In the fifth lesson, oranges and paper balls were replaced by football-sized clear plastic spheres from the Lénárt sphere set. Children worked in pairs with ine sphere for each pair to draw. We asked them to draw on the sphere with colored markers, covering the entire surface. (Figure 10) They enjoyed the task and created beautiful and varied drawings. It was interesting to note that many children attempted to build a globe without being instructed to do so.

Figure 10

Drawings on the Lénárt sphere

The transparent sphere offered a new way to grasp the meaning of opposite points. Each pair of children drew two opposite points on the sphere and then looked across the sphere at the two points, respectively. (Figure 11) They found this game very interesting and illustrative.

Figure 11

Experiencing the opposite points on the Lénárt sphere

In the sixth lesson, we took globes to approach spherical geometry from a different point of view. (Figure 12)

Figure 12

Experiencing the opposite points on the globe

We asked the children to study the globe and look for anything they found interesting or surprising. We asked them to find the North and South Poles and the Equator. (Figure 12). The two Poles were easier to find. The Equator was more difficult to locate, but in the end everyone managed it, sometimes with the teacher's help. We discovered that the Equator is a straight spherical line.

In the rest of the lesson, children were asked to locate Budapest, Rome, Madrid and Beijing on the globe and measure distances between them. Finally we discussed which city is the nearest to and farthest from Budapest.

In the seventh and final lesson, we returned to the sphere. We examined the straight line and a circle on the plane and on the sphere. Each child received a paper ball and a piece of paper. First, they drew a dot on the ball and on the paper. We asked the children if there was a difference between the dots. They then drew another dot on the paper and tried to connect the two dots along the shortest route. Extend it in both directions as far as possible.

Could you extend this line further? Everyone said yes. Now we asked the same question about two dots on the paper ball. Connect them along the shortest route on the ball and extend them as far as possible. (Figure 13) They concluded that the line returned to the starting point and could not be extended indefinitely, in contrast with the line on a piece of paper.

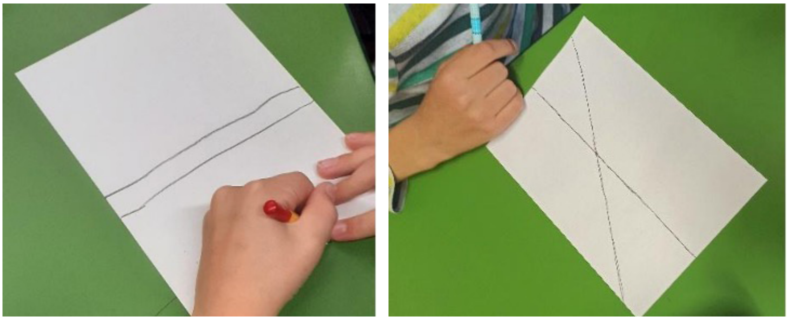
Figure 13

Drawing a line on the paper ball

We then asked the children whether the line drawn on a piece of paper was a circle. Based on their prior knowledge, they determined that it was obviously a straight line, not a circle. We then asked them to draw a circle on a piece of paper and locate its center. We then considered a straight line on a sphere (a spherical great circle) and asked the children whether this line was a circle. Several of them replied that the spherical straight line must be a circle because there is a point on the sphere from which all the points of the line are at the same distance. They correctly noticed that there are two such points on the sphere.

Figure 14

Possible location of straight lines



In the last part of the lesson we dealt with two straight lines and their mutual positions (Figure 14). Children drew two straight lines on a flat sheet and tried to determine how many points they might have in common.

Post-test after the lessons

Our research focused on whether spherical geometry can be taught in lower grades. We could not conduct a preliminary test before the experiment, as the participating children did not have knowledge of spherical geometry. In addition to the small size of the sample, this factor also contributed to the limitations of the results.

The spherical geometry post-test consisted of 30 items. It referred to the knowledge and concepts we discussed in the spherical geometry classes. It was an indicator whether spherical geometry could be taught in lower classes. This test was written by 24 students, and consisted of five subtests.

The first subtest examined students' orientation on the sphere, including their knowledge about the opposite points, the number of sides and vertices of the shapes created by dividing the spherical surface into 4 and 8 parts, and drawing circles on the plane and on the spherical surface. For example: What did we draw on the sphere? (Figure 15)

Figure 15
What did we draw on the sphere?

The second subtest measured thinking about the plane and the sphere, i.e. how much the student uses his knowledge about the plane when dealing with the sphere. The subtest asked about the number of regions adjacent to the biangle and the triangle on the sphere, as well as the properties of the circles on the plane and the sphere. (For example: What is a circle on the plane? Can you explain it?) This subtest includes items in which the planar way of thinking is a hindrance rather than an advantage on a sphere.

The third subtest covers topics in which thinking on the plane can help the student think on the sphere, as with the properties of opposite points or the triangle on both surfaces. (For example: given a point on a plane and a sphere, can you find the point furthest from the first? Or can you draw a shape on paper that has three vertices and three straight sides?)

In the fourth subtest, students had to create a geometric concept or shape on their own. The subtest involved constructing the planar equivalent of a spherical biangle and the center of a planar circle. (For example: Can you draw a shape with two vertices and two straight sides on paper?)

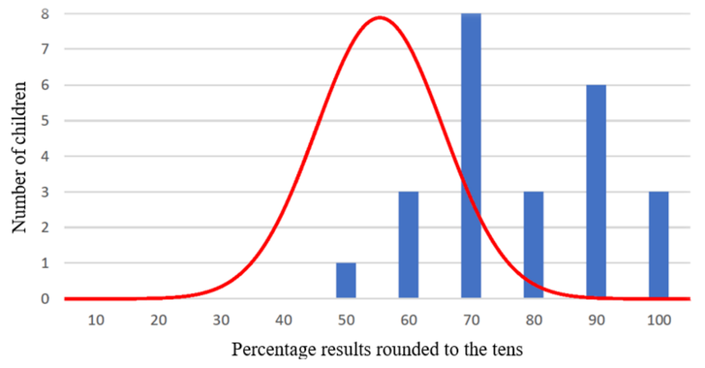
The fifth subtest checked how well students remembered the concepts they had learned. The subtest asked about the number of spherical straight lines connecting two opposite points, as well as the shapes of spherical biangles and spherical triangles. (For example: What shapes do we see on the cantaloupe. (Figure 4) Do you remember the names of the shapes?)

Presentation of test results

The result achieved on the whole test can be considered good, as the students achieved an average result of 77% after the lessons. Examining the distribution curve (Figure 16), we can see a shift to the right compared to the normal distribution (marked in red in the figure). In spherical geometry classes, students with weaker abilities also developed more intensively. The weakest test result was 50%.

Figure 16

Results of the whole test

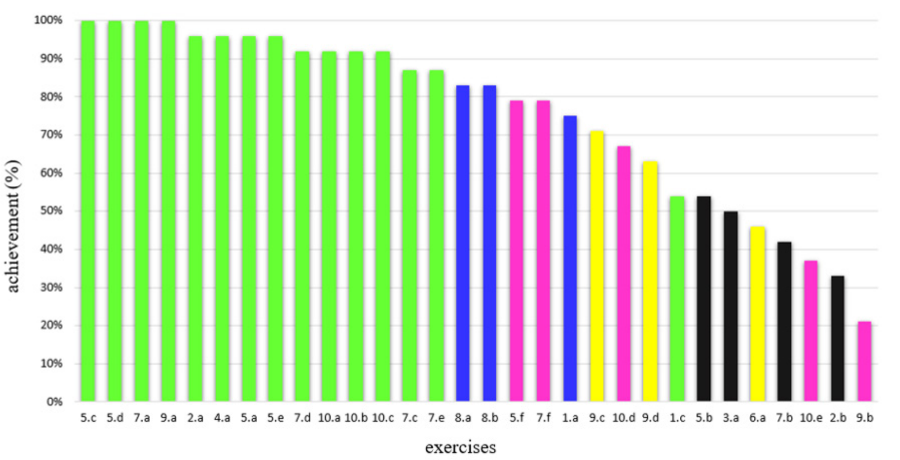


We can see a difference between the results achieved by the students in certain areas of development, i.e. in what they were able to master. These differences can be observed by examining the results of the subtests. The results of the first subtest were the best (items marked in green in Figure 17). The results show that the students are correctly oriented on the sphere, they understand the concepts, but at the same time they prefer to use their own wording instead of the technical terms on the sphere. This result is not surprising, since these concepts were encountered most often in spherical geometry classes and these concepts proved to be the most conceptual and the easiest to learn. The students performed significantly less well, but still well, on the third subtest (items marked in blue in Figure 17), which examined the "same cases" of the plane and the sphere. The good performance here is not surprising either, since in the case of this subtest, plane and spherical thinking work similarly and help each other. Clear differences were found in students' achievements in particular areas of development. These differences can be observed by examining the subtest results.

The results of the first subtest (marked in green in Figure 17) were the best, showing that students understood the relevant concepts and were correctly oriented on the spherical surface. At the same time, children prefer to use their own wording instead of the generally accepted technical terms. This result is not surprising because the sphere concepts were the most frequently used terms in the sphere classes. Students performed significantly worse, but still well, on the third subtest (marked in blue in Figure 17). The subtest examined cases in which the plane and a sphere behaved similarly. Again, the good result could be expected because this subtest covered cases of similar logic on the plane and on the sphere. Planar and spherical thinking work similarly and help each other in these cases. Again, the decrease in students' performance is also significant in the fourth subtest (marked in yellow in Figure 17). This can be explained by the fact that the students had to create and draw a figure by themselves, for example the plane equivalent of the spherical biangle. This is by no means an easy task despite the fact that we talked about it a lot with the students. An additional, but not significant, difference can be observed in the second and fifth subtests (marked in purple and black, respectively, in Figure 17). The reason may be that in the case of the second subtest, ordinary logic in plane geometry makes orientation on a sphere difficult, because the question concerned differences between the plane and the sphere. For the fifth subtest, the explanation may be that this test measured lexical knowledge that was not yet expected of the students' age group.

Figure 17

Percentage distribution of exercises



green: subtest 1; purple: subtest 2; blue: subtest 3; yellow: subtest 4; black: subtest 5

Regarding the level of the solutions, students solved the first subtest the best, around 90% or higher (exercises marked in green in Figure 17). Students were expected to recognize and draw relatively simple, well-practiced concepts. One single exercise produced significantly worse results when asked to name "opposite points." As in the lessons, in the post-test the children avoided the term "opposite point" and continued to use the terms "North Pole", "South Pole" and "furthest points".

Students coped slightly worse (50–80%) with exercises in which concepts had to be precisely named, or planar and spherical thinking interfered with each other. Albeit in different terms, exercises 5.b and 3.a asked for the name of the spherical biangle. Several students wrote "spherical petal" or "spherical ship" which they had already used in class. Far fewer of them gave the exact name "spherical biangle." Similarly, exercise 6.a asked if it is possible to draw a shape on paper with two vertices and two straight sides. (In other words, is there a planar equivalent of a spherical biangle?) The lower score was to be expected since this task had already proven to be difficult in the classroom. Planar and spherical thinking are controversial in this case.

Students performed worst on exercises 7.b, 10.e, 2.b and 9.b. For 7.b, this result is surprising because spherical and planar thinking should help each other when the exercise asks about the shape of a spherical triangle. The lower score is even more surprising because children performed significantly better on exercises 7.d and 7.e, which also relate to the concept of a spherical triangle, the number of its vertices and sides.

The lower score for task 2.b is also surprising because the students expressed in class both verbally and in drawings that two opposite points could be connected by an infinite number of straight lines. The wrong answer could have been because the children had to draw the spherical figure on a flat sheet of paper, on which the spherical straight lines could not be represented correctly. If the children had been allowed to draw on a real sphere, we would probably have received many more correct answers.

The lower performance on exercises 10.e and 9.b can be explained by the fact that the children were asked to formulate more difficult, abstract concepts in their own words. In exercise 10.e, the children had to express the difference between the great circle (i.e. the straight line on the sphere) and the spherical circle. This is a difficult question indeed. One student wrote that he knew the difference between the two shapes, but he could not articulate it. His answer is consistent with the higher proportion of responses to ex. 10.d. Several students knew that there was a difference between the two shapes, but they found it much more difficult to put this difference into words. Nevertheless, it is very encouraging that several children formulated the difference between the two shapes as "the spherical great circle is longer than the spherical circle".

Exercise 9.b ("What makes a circle a circle?") proved to be the most difficult for the students, despite the fact that the question referred to the planar circle. At the same time, several students (remember that they are 3rd graders!) gave the mathematically perfect answer, “Because all its parts are the same distance from its center.”

Conclusion

On the basis of our experiences in three-graders’ classes, we can say that spherical geometry is a worthy counterpart to plane geometry from a mathematical and pedagogical point of view.

Spherical geometry has a very important advantage over plane geometry, namely, it is built on a finite surface, in contrast with the infinite plane.

Understanding the spherical concepts is greatly facilitated by direct experimentation on real spheres, such as oranges, paper balls and Lénárt (n.d.) spheres in the present case.

Comparing the plane and the sphere also educates the students to ask questions from each other and from their teachers, to investigate and experiment on their own, and to compare their thoughts with those of others.

Many basic concepts of spherical geometry are also included in the material of another subject, geography. We tried to present this connection at the elementary school level in the lesson about the globe.

It is known that lower elementary school children are characterized by the first two levels of the van Hiele model (Herendiné Kónya, 2003). Grades 1-2. correspond to level 0 of global recognition), while grades 3-4. to level 1of analysis (Pintér, 2013).

Our third graders, who were introduced to the basics of spherical geometry, reached Level 1 in knowledge of Euclidean geometry and Level 0 in spherical geometry. Over the course of seven lessons, the children had not yet reached Level 1 in spherical geometry, but they were able to complete the first three phases of information – guided discovery – explanation (Herendiné Kónya, 2004). The fourth phase of unguided discovery could not be fully implemented due to lack of time. Very importantly, the third phase was accomplished in each lesson, where the students explained their observations and related ideas to each other. Students really enjoyed telling each other about their experiences and thoughts on their findings in the new world of spherical geometry.

Students’ good results in the post-test proved that teaching spherical geometry content was worthwhile for this age group. Therefore, planning further research based on a larger sample should also be considered.

The results also revealed that students with lower abilities showed more intensive development in spherical geometry classes. When studying plane geometry and spherical geometry simultaneously, it becomes completely natural for the children to accept and compare the concepts of the two geometries. As one student put it, “It was interesting to see in these classes why we cannot draw the same thing on a flat sheet as on the sphere.”

During the study of the globe, it became clear that learning spherical geometry can contribute to easier orientation on the globe and a better understanding of basic geographical concepts.

Finally we give a few thoughts regarding the continuation of the research. Based on the theoretical summary above, it can be concluded that the development of spatial thinking has a positive effect on other areas of mathematics as well. The most related field is plane geometry, because the comparison not only leads to a better understanding of spherical geometry and the development of spatial perception, but strengthens the knowledge of plane geometry.

It would also be interesting to examine, relying on the literature, whether the teaching of spherical geometry has an effect on the ability to count and handle arithmetical operations with numbers. It would be worthwhile to take a pre- and post-test and analyze the effect of spherical geometry from this point of view.

Another long-term goal could be to try out the extended version of the above experimental material with fourth-grade students and observe the developmental effect of spherical geometry in other areas of spatial geometry and mathematics. This investigation would be particularly interesting with the third-graders in the present experiment in order to examine their further development in higher classes of mathematics and geometry.

Acknowledgement

The current research was supported by the MTA-SZTE Metacognition Research Group.

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A gömbi geometria tanításának lehetőségei az alsó tagozatos matematika órákon

Kutatásunk arra irányult, hogy a gömbi geometria tanítható-e alsó tagozaton. Harmadik osztályos, alsó tagozatos tanulók (N=28) körében próbáltuk ki a gömbi geometria néhány egyszerű alapfogalmára épülő szemléletes és játékos geometriai tevékenységet. A gömbi geometriai tanórák során azt tapasztaltuk, hogyha a gyerekek egy időben ismerkednek meg a síkgeometriával és a gömbi geometriával, akkor számukra teljesen természetessé válik a síkgeometriai és a gömbi geometriai fogalmak elfogadása és összehasonlítása. A földgömb tanulmányozása során egyértelműen látszott az is, hogy a gömbi geometria tanulása hozzájárulhat a földgömbön való könnyebb tájékozódáshoz és a földrajzi alapfogalmak jobb megértéséhez. A tanulók utótesztben elért jó eredményei azt igazolták, hogy az adott alsó osztályban érdemes volt gömbi geometriát tanítani, és ennek alapján további, nagyobb mintán végzett kutatások igénye is felmerülhet. Az eredményekből az is láthatóvá vált, hogy a gömbi geometria tanórákon még a gyengébb képességű tanulók is intenzívebben fejlődtek.

Kulcsszavak: matematika, gömbi geometria, alsó tagozat

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