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Notes on localizing Craig's interpolation theorem

1. Introduction and philosophical motivation

Craig's interpolation theorem has become a central logical property that expresses a deep connection between the syntax and semantics of first-order logic, and has been used to prove other important theorems of first-order logic, such as Beth definability or Robinson's joint consistency theorem. Craig's theorem has been generalized and extended in several ways, and interpolation properties of general logics have been intensively studied in the literature ever since. Craig's original result, in a somewhat informal way, is that whenever an implication $\varphi \rightarrow \psi$ is a tautology of first-order logic, then there is a formula χ such that $\vDash \varphi \rightarrow \chi$ and $\vDash \chi \rightarrow \psi$ hold, and all the non-logical predicate symbols of χ are both in φ and ψ. In his 2008 essay, Craig writes about the origins of his theorem:

> Although I was aware of the mathematical interest of questions related to elimination problems in logic, my main aim, initially unfocused, was to try to use methods and results from logic to clarify or illuminate a topic that seems central to empiricist programs: In epistemology, the relationship between the external world and sense data; in philosophy of science, that between theoretical constructs and observed data.[1]

To clarify the picture, consider a scientific theory (say, some part of physics or biology or the like) that is axiomatized in first-order logic. Such an axiomatization may use predicates that refer to theoretical constructs, and predicates that refer to observational data. To simplify exposition, suppose that the axiomatization in question is finite, or what amounts to the same, using conjunctions, it is one formula φ. If ψ is an observational consequence of φ (and so ψ is expressed in the observational vocabulary of the theory) then by Craig's theorem one obtains an axiomatization of the observational consequences ψ by means of an interpolant formula χ in which only symbols for the observational vocabulary occur.[2] In effect, Craig's result gives logical tools to eliminate the theoretical terms. According to Putnam: "This had led some authors to advance the argument that, since the purpose of science is successful prediction, theoretical terms are in principle unnecessary."[3]

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1 Craig 2008, 8.
2 Mancosu 2008.
3 Putman 1965.

This issue of elimination of theoretical terms from a Carnapian reconstruction of science is a central topic in Hempel and Demopoulos.[4] Both papers criticize logical reconstructions that identify the theoretical content of a scientific theory with logical truth, and the systemization provided by Craig's re-axiomatization. In this paper, we introduce the *modelwise interpolation property* of a logic (not necessarily first-order), which states that whenever $\vDash\varphi\rightarrow\psi$ holds, then one can find an interpolant formula in every model, that is, the interpolant formula in Craig interpolation may vary from model to model. In order to make sense of this notion, we have to work with logics that are semantically defined; e.g., a notion of model should be built in the definition of the logic. We have three main motivations as follows.

1. Scientific theories are sometimes axiomatized by logics other than classical first-order logic, for example, modal logic is used to axiomatize relativity theory.[5] Such logics may or may not have the Craig interpolation property. If the logic we make use has no Craig interpolation but turns out to have the modelwise interpolation property, and our scientific theories are formulated in this logic and evaluated in a model (see the next item), then changing our background logic from first-order logic to this new logic still allow us to carry out arguments inside models similar to Craig's.
2 There is a tension between global and local approaches and it can be argued that when it comes to scientific theories, global truth (in the logical reconstruction, the tautologies of the logic) might not be as informative as local truth (which is truth with respect to specific models). Unfolding this topic would lead too far, and we refer to Rus's PhD thesis instead.[6] We also mention that most of the cases when it comes to logical reconstructions of physical theories, the predictions take place in specific models where e.g. the real numbers and similar mathematical objects are available. It is a well-known consequence of the Löwenheim–Skolem theorems that the structure of real numbers is not axiomatizable in first-order logic, therefore that sort of axiomatic approach suggested by Craig might not be feasible, at least not directly.[7]
3 Finally, there is a tradition in algebraic logic to study local versions of classical theorems of logics, e.g. one defines the notion of local explicit definition with respect to weak Beth definability property.[8] Studying such localized properties

4 Demopoulos 2008 and Hempel 1958.
5 See Molnár 2013. For axiomatizing relativity theory see in general Madarász – Németi – Székely 2006 and Székely 2009.
6 David-Rus 2009.
7 For more details, see Székely 2009.
8 Andréka – Németi – Sain 2001, Chapter 6.

can shed light on the connections between syntax and semantics, as well as can serve as dividing lines when comparing different logics.

In this paper, we focus on one main example, which is Difference logic. We show that while it lacks the standard Craig interpolation property, still it has the modelwise interpolation property. This gives us two technical applications: we show that difference logic has the local Beth definability property and a Robinson joint consistency type of property. In the last section we make a detour in modal logic and draw some further consequences.

2. A formal treatment

In order to abstractly formulate the modelwise interpolation property we have to use a framework for logics in which there is a built-in notion of models.[9] By a logic, we understand a tuple $\mathcal{L}$=⟨F, M, ⊨⟩, where F is a set of formulas generated by a set P of atomic formulas using logical connectives; M is an abstract class of models; and the consequence relation ⊨ is a relation of the type $\vDash \subset M \times F$. We assume that there are two distinguished connectives: a binary → denoting implication, and a constant ⊥ standing for falsity. For a formula $\alpha \in F$, Voc(α) denotes the set of atomic formulas occurring in α. As it is standard in logic we extend the consequence relation ⊨ to a relation in between (sets) of formulas: For $\Gamma, \{\varphi\} \subseteq F$ we write $\Gamma \vDash \varphi$ if whenever $\mathfrak{M} \vDash \Gamma$ for a model $\mathfrak{M} \in M$, then $\mathfrak{M} \vDash \varphi$ as well.

Definition 2.1. We say that the logic $\mathcal{L}$ = ⟨F, M, ⊨⟩ has the *modelwise interpolation property* if for every formula $\varphi, \psi \in F$, if $\vDash \varphi \to \psi$, then for all models $\mathfrak{M} \in M$ there exists $\chi \in F$ with $\mathrm{Voc}(\chi) \subseteq \mathrm{Voc}(\varphi) \cap \mathrm{Voc}(\psi)$ such that $\mathfrak{M} \vDash \varphi \to \chi$ and $\mathfrak{M} \vDash \chi \to \psi$.

Note that it is crucial for the definition of the modelwise interpolation property to have a notion of model built in the definition of the logic L, therefore the definition cannot be applied to purely syntactically given logical calculi.

Recall that the Craig interpolation property is the property that whenever $\varphi, \psi \in F$, if $\vDash \varphi \to \psi$, then there exists $\chi \in F$ with $\mathrm{Voc}(\chi) \subseteq \mathrm{Voc}(\varphi) \cap \mathrm{Voc}(\psi)$ such that $\vDash \varphi \to \chi$ and $\vDash \chi \to \psi$. The modelwise interpolation property thus differs from Craig's interpolation in that the interpolant formula is "localized" i.e. it may vary from model to model.

9 The most straightforward choice would be to rely on the Andréka – Németi – Sain (see Andréka – Gyenis – Németi – Sain 2022, Andréka – Németi – Sain 2001, and Hoogland 1996, 2001, or Madarász 1998) approach rather than the more mainstream Blok–Pigozzi (see Block¬ – Pigozzi 1989, 1991,1994 and Czelakowski – Pigozzi 2004) framework in which the focus is rather on the relation ⊢ between sets of formulas and is missing the general notion of models. However, we try to keep the formal parts as simple as possible, therefore we make slight simplifications in the ANS-framework.

It is straightforward to see that Craig's interpolation property implies the modelwise interpolation property: a global interpolant is an interpolant in every model. However, the converse is not true: we prove that first-order logic restricted to n variables has the modelwise interpolation property but lacks the Craig interpolation property. Further, to show that modelwise interpolation is not an automatic property of a logic, we show that Łukasiewicz's logic does not have neither Craig's interpolation, nor the modelwise interpolation. Even though our definitions so far were employed for logics in a very broad sense, the examples given below are all well-studied in the literature, having further "nice" properties such as algebraizability.

Difference logic $\mathcal{L}_D$. Difference logic is a kind of modal logic and is discussed in the literature.[10] e.g. in Sain, Venema, Roorda, but see also Segerberg, who traces this logic back to von Wright. For a set P of atomic formulas, the set F is generated using the connectives {∧, ¬, ⊥, D}, where conjunction, negation and falsity are the usual, and D is a unary connective. Models are of the form

$$\mathfrak{M}=\langle W,V\rangle, \text{ where } W\neq\emptyset \text{ and } V:P\rightarrow\wp(W).$$

For a model $\mathfrak{M}$, $w\in W$ and a formula φ one defines $M,w\Vdash\varphi$ by

$$\begin{aligned}
\mathfrak{M}, w \nVdash \bot &\Leftrightarrow (\exists x \neq u)\mathfrak{M}, x \Vdash \vartheta \\
\mathfrak{M}, w \Vdash p &\Leftrightarrow w \in V(p) \\
\mathfrak{M}, w \Vdash \varphi \wedge \psi &\Leftrightarrow \mathfrak{M}, w \Vdash \varphi \text{ and } \mathfrak{M}, w \Vdash \psi \\
\mathfrak{M}, w \Vdash \neg \varphi &\Leftrightarrow \mathfrak{M}, w \nVdash \varphi \\
\mathfrak{M}, w \Vdash D\varphi &\Leftrightarrow (\exists\, v \in W\backslash\{w\})\ \mathfrak{M}, \Vdash \varphi
\end{aligned}$$

Finally, we define $\vDash$ by

$$\mathfrak{M} \vDash \varphi \Leftrightarrow (\forall\, w \in W)\ \mathfrak{M}, w \Vdash \varphi$$

Difference logic does not have the Craig interpolation property. Let us briefly recall the argument. Let $E\varphi$ abbreviate $\varphi\vee D\varphi$. The following implication is a logical validity of difference logic:

$$\vDash_{\mathcal{L}_D} (Dp \wedge D\neg p) \rightarrow (E(r \wedge \neg Dr) \rightarrow E(\neg r \wedge D\neg r))$$

10 Difference logic is studied at depth from many different aspects in Sain 1983, Segerberg 1970, de Rijke 1993, Rooda 1991, Venema 1992.

The reason is that in a model M and a world w, $w \Vdash Dp \wedge D\neg p$ implies that there are at least two other worlds not equal to w, while $E(r \wedge \neg Dr) \rightarrow E(\neg r \wedge D\neg r)$ expresses that if there is only one world satisfying r, then there must be at least two different worlds satisfying ¬r. The common vocabulary of the subformulas on the two side of the implication is empty, and it is not hard to check that neither ⊤ nor ⊥ can be global interpolant.[11] However, $\mathcal{L}_D$ has the modelwise interpolation property.

Theorem 2.2. $\mathcal{L}_D$ has the modelwise interpolation property.

Proof. Suppose $\vDash_{\mathcal{L}_D} \varphi(\vec{p},\vec{q}) \rightarrow \psi(\vec{q},\vec{r})$ is a logical validity, where the formulas φ and ψ use the atomic formulas $\vec{p}$, $\vec{q}$ and $\vec{r}$ as denoted. We need to find an interpolant formula using the atomic formulas $\vec{q}$ only. Write $\vec{q}=\langle q_0, \dots, q_{n-1}\rangle$ and $\vec{p}=\langle p_0, \dots, p_{m-1}\rangle$. Two worlds, $v, w \in W$ are said to be $\vec{q}$-equivalent ($v \sim w$ in symbols) if for all $i<n$ we have

$$\mathfrak{M}, v \Vdash q_i \Leftrightarrow \mathfrak{M}, w \Vdash q_i$$

Claim 2.3. If $\mathfrak{M}, v \Vdash \varphi$ and $w \sim v$, then $\mathfrak{M}, w \Vdash \psi$.

Proof. Assume $\mathfrak{M}, v \Vdash \varphi$ and define a new model $\mathfrak{M}' = \langle W, V' \rangle$ on the same set of possible worlds as follows. For a world $u \in W$ let us use the notation

$$u' = \begin{cases} v, & \text{if } u=w \\ w, & \text{if } u=v \\ u, & \text{if } u \neq v, u \neq w \end{cases}$$

that is, we exchange v with w but keep everything fixed. Define the new evaluation V' by $V'(q_i)=V(q_i)$, $V'(r_i)=V(r_i)$ and

$$V'(p_i) = \{u' : u \in V(p_i)\}.$$

Lemma 2.4. For any formula $\vartheta(\vec{p},\vec{q})$ and world $u \in W$ we have

$$M, u \Vdash \vartheta \Leftrightarrow M', u' \Vdash \vartheta.$$

Proof. Induction on the complexity of ϑ.

11 For more details, see Conradie 2002.

- For atomic propositions q_i: As $V'(q_i) = V(q_i)$, if $u \neq v$ and $u \neq w$, then $u = u'$ and thus the statement holds. For $u = v$ or $u = w$ we obtain the result by assumption $v \sim w$.
- For atomic propositions p_i the statement follows directly from the definition of V': $\mathfrak{M}, u \Vdash p_i$ if and only if $\mathfrak{M}', u' \Vdash p_i$.
- For the Boolean combinations the induction is straightforward.
- For formulas of the form $D\vartheta$: Assume (inductive hypothesis) that the statement holds for ϑ. Then

$$\begin{aligned}
\mathfrak{M}, u \Vdash D\vartheta &\Leftrightarrow (\exists x \neq u)\mathfrak{M}, x \Vdash \vartheta \\
&\Leftrightarrow (\exists x' \neq u)\mathfrak{M}', x' \Vdash \vartheta \\
&\Leftrightarrow (\exists x' \neq u')\mathfrak{M}', x' \Vdash \vartheta \\
&\Leftrightarrow \mathfrak{M}', u' \Vdash D\vartheta
\end{aligned}$$

□

Applying the lemma to v and φ we obtain $\mathfrak{M}', w \Vdash \varphi$. As $\vDash_{\mathcal{L}_D} \varphi \to \psi$ holds we get $\mathfrak{M}'$, $w \Vdash \psi$. But note that V and V' coincide on the elements of $\vec{q}$ and $\vec{r}$, therefore $\mathfrak{M}$, $u \Vdash \psi$ if and only if $\mathfrak{M}, u' \Vdash \psi$. It follows $\mathfrak{M}, w \Vdash \psi$ completing the proof of the claim.

□

In what follows we use the notation $q^1 = q$ and $q^0 = \neg q$. For $v \in W$ we write

$$\chi_v = \bigwedge_{i<n} q_i^{\varepsilon_i}$$

where

$$\varepsilon_i = \begin{cases} 1 & \text{if } \mathfrak{M} \vDash q_i[v] \\ 0 & \text{otherwise"} \end{cases}$$

By the claim above for each v for which $\mathfrak{M}, v \Vdash \varphi$ holds, the equivalence class $v/\sim$ is a subset of $\{u \in W : \mathfrak{M}, u \Vdash \psi\}$. As $\vec{q}$ is finite, there are only finite many $\sim$ equivalence classes. Let $v_0, \ldots v_l$ be representative elements of all the different equivalence classes such that $\mathfrak{M}, v_i \Vdash \varphi$ and write

$$\chi = \bigvee_{i<l} \chi_{v_i}$$

Then $\mathfrak{M} \vDash \varphi \to \chi$ and $\mathfrak{M} \vDash \chi \to \psi$, that is χ is a desired interpolant formula in $\mathfrak{M}$.

□

Łukasiewicz's logic Ł. Consider the 3-element algebra

$$\mathfrak{A} = \langle \{0, 1/2, 1\}, \wedge, \vee, \neg, \to, 1 \rangle,$$

where the operations are given by

$$x \wedge y = \min\{x, y\}, \ \ x \vee y = \max\{x, y\},$$
$$\neg x = 1 - x, \quad x \to y = \min\{1, 1 - x + y\}.$$

Łukasiewicz's logic Ł is defined as follows. The logical connectives are the usual $\wedge$, $\vee$, $\neg$, $\to$, $\top$. If P is a set of propositional variables, then the set of formulas F is generated by P using the connectives. Write $\mathcal{F}$ for the absolutely free formula algebra $\mathcal{F} = \langle F, \wedge, \vee, \neg, \to, \top\rangle$. The class of models is

$$M = \{h: \mathcal{F} \to \mathfrak{A} : h \text{ is a homomorphism}\}.$$

In a model $h \in M$, $h \vDash \varphi$ holds if $h(\varphi) = 1$. The definition of logical validity is then

$$\vDash_{\text{Ł}} \varphi \leftrightarrow (\forall h \in M) h(\varphi) = 1$$

It is easy to check via truth tables that the implication

$$\vDash_{\text{Ł}} p \wedge \neg p \to q \vee \neg q$$

holds for any propositional variables $p, q \in P$. Every formula in the empty vocabulary is equivalent to either $\top$ or $\neg\top$ $(=\bot)$. However, in the model where both p and q are evaluated to 1/2 neither $\top$ not $\bot$ can be an interpolant. Therefore Ł does not have the modelwise interpolation property, and thus it does not have the Craig interpolation property either.

3. Two corollaries

The local Beth property of a logic $\mathcal{L}$ states that every implicitly definable relation is locally explicitly definable, that is, the explicit definition may vary from model to model.[12] To be more precise, let $\mathcal{L} = \langle F, M, \vDash\rangle$ be a logic, write F^P to denote the set of formulas of the logic $\mathcal{L}$ that are generated by the propositional letters P, that is, $F^P = \{\varphi \in F: Voc(\varphi) \subseteq P\}$, and let $\leftrightarrow$ be a distinguished binary connective. For a set of propositional letters R let R' be a disjoint copy of R and for $\Sigma \subseteq F^R$ we write Σ' to denote the formulas obtained from Σ be replacing each $r \in R$ by the corresponding $r' \in R'$. We say that $\Sigma \subseteq F^{P \cup R}$ *defines* R *implicitly* in terms of P if $\Sigma \cup \Sigma' \vDash r \leftrightarrow r'$ for every $r \in R$. Further, Σ *defines* R

12 The concept was introduced in Andréka – Sain – Németi 2001, Definition 6.9.

locally explicitly in terms of P if for every model $\mathfrak{M} \vDash \Sigma$, for all $r \in R$ there is $\varphi_r \in F^P$ such that $\mathfrak{M} \vDash r \leftrightarrow \varphi_r$. That is, the usual explicit definition may vary from model to model.

We show that the modelwise interpolation property implies the local Beth definability property for a wide range of logics. In what follows, we work with logics that extend classical propositional logic in the sense that the connectives $\wedge$ and $\rightarrow$ are available and work in the usual way. The logic L is said to be consequence compact if for every $\Gamma,\{\varphi\} \subseteq F$, if $\Gamma \vDash \varphi$, then there is a finite subset $\Gamma_0 \subseteq \Gamma$ such that $\Gamma_0 \vDash \varphi$. $\mathcal{L}$ is conjunctive if for any $\varphi, \psi \in F$ we have

$$\{\vartheta: \varphi, \psi \vDash \vartheta\} = \{\vartheta: \varphi \wedge \psi \vDash \vartheta\}.$$

We say that L has deduction theorem if for all $\varphi, \psi, \vartheta \in F$ we have

$$\varphi, \psi \vDash \vartheta \quad \text{if and only if} \quad \varphi \vDash \psi \rightarrow \vartheta.$$

Theorem 3.1. Suppose $\mathcal{L}$ is consequence compact, conjunctive, and has deduction theorem. If $\mathcal{L}$ has the modelwise interpolation property, then it has the local Beth definability property.

Proof. The proof is standard. Suppose that $\Sigma \subseteq F^{P\cup\{r\}}$ defines r implicitly, that is

$$\Sigma \cup \Sigma' \vDash r \leftrightarrow r'.$$

By consequence compactness and conjunctiveness there is a formula φ such that

$$\varphi, \varphi' \vDash r \leftrightarrow r'.$$

By deduction and conjunctiveness

$$\vDash (\varphi \wedge \varphi') \rightarrow (r \leftrightarrow r').$$

For any model $\mathfrak{M}$, by the modelwise interpolation property, there is an interpolant formula $\vartheta_{\mathfrak{M}} \subseteq F^P$ such that

$$\mathfrak{M} \vDash \varphi \wedge r \rightarrow \vartheta_{\mathfrak{M}}, \quad \text{and} \quad \mathfrak{M} \vDash \vartheta_{\mathfrak{M}} \rightarrow (\varphi' \rightarrow r'),$$

hence

$$\mathfrak{M} \vDash \varphi \rightarrow (r \leftrightarrow \vartheta_{\mathfrak{M}}).$$

Using deduction, for every $\mathfrak{M} \vDash \Sigma$ one has $\mathfrak{M} \vDash r \leftrightarrow \vartheta_{\mathfrak{M}}$, that is, Σ locally explicitly defines r.

□

Corollary 3.2. Difference logic $\mathcal{L}_D$ has the local Beth definability property.

Proof. Combine Theorems 2.3 and Theorem 3.1.

□

Next, using the example of difference logic $\mathcal{L}_D$, we show that the modelwise interpolation property can imply interesting global properties, in our example below this is a weak version (see the comments after the theorem) of Robinson's joint consistency theorem.

Theorem 3.3. Consider difference logic $\mathcal{L}_D$ and suppose P_1 and P_2 are disjoint. Assume $\Sigma_1 \subseteq F^{P_1}$ and $\Sigma_2 \subseteq F^{P_2}$ are consistent. Then $\Sigma_1 \cup \Sigma_2 \subseteq F^{P_1 \cup P_2}$ is consistent as well.

Proof. We more or less follow the standard proof that uses global interpolation and make some modifications that allows us to refer to the modelwise version of interpolation. Recall that $\mathcal{L}_D$ has the modelwise interpolation property by Theorem 2.3.

By way of contradiction, suppose that $\Sigma_1 \cup \Sigma_2$ is inconsistent. Then there are finite $\Gamma_1 \subseteq \Sigma_1$ and $\Gamma_2 \subseteq \Sigma_2$ such that for $\gamma_1 = \bigwedge \Gamma_1$ and $\gamma_2 = \bigwedge \Gamma_2$ we have $\vDash_{\mathcal{L}_D} \gamma_1 \rightarrow \neg\gamma_2$. By the modelwise interpolation property for all models $\mathfrak{M}$ there is a formula χ such that $\mathrm{Voc}(\chi) \subseteq \mathrm{Voc}(\gamma_1) \cap \mathrm{Voc}(\gamma_2)$ and $\mathfrak{M} \vDash \gamma_1 \rightarrow \chi$ and $\mathfrak{M} \vDash \chi \rightarrow \neg\gamma_2$.

Take the canonical model $\mathfrak{M}$ in the language $P_1 \cup P_2$. Let χ be the interpolant formula in the model $\mathfrak{M}$.[13] As Σ_1 is consistent, there is some maximal consistent set Δ_1 such that $\gamma_1 \in \Delta_1$, and similarly, there is a maximal consistent set Δ_2 such that $\gamma_2 \in \Delta_2$. By the Truth Lemma[14] it follows that

$$\mathfrak{M}, \Delta_1 \vdash \chi, \quad \text{and} \quad \mathfrak{M}, \Delta_2 \vdash \neg\chi.$$

But this is a contradiction as χ is formulated in the empty language $P_1 \cap P_2$.

□

In the general case, Robinson's joint consistency property would be the following statement:

> Suppose $P_0 = P_1 \cap P_2$ and $\Sigma_0 \subseteq F^{P_0}$ is a maximally consistent set of formulas. Assume $\Sigma_1 \subseteq F^{P_1}$ and $\Sigma_2 \subseteq F^{P_2}$ are consistent extensions of Σ_0. Then $\Sigma_1 \cup \Sigma_2 \subseteq F^{P_1 \cup P_2}$ is consistent.

13 See de Rijke 1993, Definition 3.3.15.
14 de Rijke 1993, Lemma 3.3.17.

In the theorem above, we took Σ_0 to be the empty set. Our statement in Theorem 3.3 could be strengthened by requiring Σ_0 to be canonical (i.e. the canonical model of Σ_0-consistent sets exists and satisfies the Truth lemma). We do not pursue such a generalization in this paper.

4. Detour into modal logics

Ever since Craig proved his original interpolation for first-order logic, the property has been widely studied in modal and intermediate logics. From the semantic perspective the main direction was algebraic in nature via amalgamation properties of certain classes of modal algebras.[15] We do not pursue the algebraic approach here, rather, we focus on the model theoretic connections. For this approach, Marx introduced interesting model theoretic conditions that are useful to prove or disprove whether a canonical modal logic has the Craig interpolation.[16] Later, ten Cate proposed similar requirements for elementary classes.[17] Continuing this track, we establish a sufficient condition for a class having the *modelwise interpolation property.* Before doing so, we recall some basic notions from modal logics.

The standard unimodal language is defined by the following grammar as:

$$p|\top|\neg\varphi|(\varphi\vee\psi)|\Diamond\varphi)$$

where p is a propositional letter and $\Box$ abbreviates $\neg\Diamond\neg$. A set Λ of modal formulas is called *normal modal logic,* if it contains all propositional tautologies and is closed under *modus ponens, uniform substitution and modal generalization,* moreover it contains the axioms:

$$K:=\Box(p\rightarrow q)\rightarrow(\Box p\rightarrow\Box q)$$
$$\text{Dual}:=\Diamond p\leftrightarrow\neg\Box\neg p$$

Models for the language are tuples $\mathfrak{M}=\langle\mathfrak{F},V\rangle$, where $\mathfrak{F}=\langle W,R\rangle$ is the underlying frame or structure equipped with a binary relation R, and $V\colon P\to\wp(W)$ is a valuation of the variables. Truth of a formula at $w\in W$ is defined as usual, except the case for $\Diamond$:

$$\mathfrak{M},w\Vdash\Diamond\varphi\Leftrightarrow\exists v\in W\text{ s.t. }Rwv\text{ and }\mathfrak{M},v\Vdash\varphi$$

15 Maksimova 1979, 1991.
16 Marx 1995, 1999.
17 ten Cate 2004.

We say φ is *globally satisfied* in $\mathfrak{M}$, in symbols: $\mathfrak{M} \vDash \varphi$, if for all $w \in W$, $\mathfrak{M}, w \Vdash \varphi$. By definition, φ is *valid* in $\mathfrak{F}$, notation: $\mathfrak{F} \Vdash \varphi$ if for all model based on $\mathfrak{F}$ we have $\mathfrak{M} \Vdash \varphi$. Let $\mathfrak{M}$ be a class of models, $\mathcal{K}$ be a class of frames. We introduce the notation:

$$\mathrm{Th}(\mathcal{M}) = \{\varphi\colon (\forall \mathfrak{M} \in \mathcal{M})\, \mathfrak{M} \Vdash \varphi\}$$
$$\mathrm{Mod}(\mathrm{Th}(\mathcal{M})) = \{\mathfrak{M}\colon \mathfrak{M} \Vdash \mathrm{Th}(\mathcal{M})\}$$
$$\mathrm{Th}(\mathcal{K}) = \{\varphi\colon (\forall \mathfrak{F} \in \mathcal{K})\, \mathfrak{F} \Vdash \varphi\}$$
$$\mathrm{Mod}(\mathrm{Th}(\mathcal{K})) = \{\mathfrak{F}\colon \mathfrak{F} \Vdash \mathrm{Th}(F)\}$$

Note that $\mathrm{Th}(\mathcal{K})$ is a (normal) modal logic defined in the above sense and is called the logic generated by $\mathcal{K}$. We do not recall further standard notions, such as *bounded morphism, generated substructure and bisimulation.*[18] Our key notion will be the following:

Definition 4.1. A bisimulation product of a set of frames $\{\mathfrak{F}_i\colon i \in I\}$ is a subframe $\mathfrak{G}$ of the Cartesian product $\prod_i \mathfrak{F}$ such that for each $i \in I$, the natural projection $\pi_i\colon \mathfrak{G} \to \mathfrak{F}_i$ is a surjective bounded morphism.

Fact 4.2. Let H be a submodel of the product $\mathfrak{F} \times \mathfrak{G}$. Then $\mathfrak{H}$ is a bisimulation product of $\mathfrak{F}$ and $\mathfrak{G}$ iff the domain of H is a total frame bisimulation between $\mathfrak{F}$ and $\mathfrak{G}$.

By a total bisimulation between $\mathfrak{F}$ and $\mathfrak{G}$, we mean a bisimulation whose domain is the universe of $\mathfrak{F}$ and the range is the universe of $\mathfrak{G}$. We say that a class $\mathcal{K}$ of frames is *closed under bisimulation products* if for all $\mathfrak{F}, \mathfrak{G} \in \mathcal{K}$, all bisimulation products of $\mathfrak{F}$ and $\mathfrak{G}$ are in $\mathcal{K}$.

Theorem 4.3. Let $\mathcal{K}$ be an elementary class of frames closed under generated subframes and bisimulation product. Then the modal logic generated by $\mathcal{K}$ has the Craig interpolation property.[19]

Now we are going to state and prove its *modelwise* analogue. For this, we need the following concept:

Definition 4.4. A class $\mathcal{K}$ is closed under modelwise bisimulation products if for all model $\mathfrak{M}$ based on $\mathcal{K}$ and all models $\mathfrak{N}, \mathfrak{N}' \in \mathrm{Mod}(\mathrm{Th}(\mathfrak{M}))$ every bisimulation products between $\mathfrak{N}$ and $\mathfrak{N}'$ are in $\mathcal{K}$.

Theorem 4.5. If $\mathcal{K}$ is closed under modelwise bisimulation products, then the modal logic generated by $\mathcal{K}$ has the modelwise interpolation property.

18 For more details see Blackburn – de Rijke – Venema 2001.
19 The statement is proved in ten Cate 2004.

Proof. Suppose $\mathcal{K} \vDash \varphi \to \psi$, and by contradiction assume there is some model $\mathfrak{M}$ based on a frame in $\mathcal{K}$ such that $\mathfrak{M} \nvDash \varphi \to \theta$ or $\mathfrak{M} \nvDash \theta \to \psi$, for all formula ϑ with $\mathrm{Voc}(\vartheta) \subseteq \mathrm{Voc}(\varphi) \cap \mathrm{Voc}(\psi)$. Define the set

$$\mathrm{Cons}(\varphi) = \{\vartheta : \mathfrak{M} \vDash \varphi \to \vartheta, \mathrm{Voc}(\vartheta) \subseteq \mathrm{Voc}(\varphi) \cap \mathrm{Voc}(\psi)\}$$

Claim 4.6. There is a model $\mathfrak{N}^+ \in \mathrm{Mod}(\mathrm{Th}(\mathfrak{M}))$ with a world w such that $\mathfrak{N}^+, w \vDash \mathrm{Cons}(\varphi) \cup \{\neg\psi\}$.

> Observe that every finite subset of $\mathrm{Cons}(\varphi) \cup \{\neg\psi\}$ is satisfiable in $\mathrm{Mod}(\mathrm{Th}(\mathfrak{M}))$. Otherwise there are some $\vartheta_1, \dots \vartheta_n, \in \mathrm{C\ ons}(\varphi)$ such that for all $\mathfrak{N} \in \mathrm{Mod}(\mathrm{Th}(\mathfrak{M}))$ we have $\mathfrak{N} \vDash \bigwedge \vartheta_i \to \psi$. Then $\bigwedge\vartheta_i$ is an interpolant in $\mathfrak{M}$, contrary to the assumption. By standard results, there is an ultraproduct $\mathfrak{N}^+$ and w such that $\mathfrak{N}^+, w \vDash \mathrm{Cons}(\varphi) \cup \{\neg\psi\}$, also $\mathrm{Th}(\mathfrak{M}) \subseteq \mathrm{Th}(\mathfrak{N}^+)$, hence $\mathfrak{N}^+ \in \mathrm{Mod}(\mathrm{Th}(\mathfrak{M}))$.

Now, we define the set

$$\Sigma = \{\vartheta : \mathfrak{N}^+, w \Vdash \vartheta, \mathrm{Voc}(\vartheta) \subseteq \mathrm{Voc}(\varphi) \cap \mathrm{Voc}(\psi)\}$$

Claim 4.7. There is a model $\mathfrak{N}^* \in \mathrm{Mod}(\mathrm{Th}(\mathfrak{M}))$ with a world v such that $\mathfrak{N}^*, v \Vdash \Sigma \cup \{\varphi\}$.

> Again, every finite subset of $\Sigma \cup \{\varphi\}$ is satisfiable in $\mathrm{Mod}(\mathrm{Th}(\mathfrak{M}))$. Otherwise there are some $\vartheta_1, \dots, \vartheta_n \in \Sigma$ such that for all $\mathfrak{N} \in \mathrm{Mod}(\mathrm{Th}(\mathfrak{M}))$ we have $\mathfrak{N} \vDash \varphi \to \neg\bigwedge\vartheta_i$. Then again $\bigwedge\vartheta_i \in \mathrm{Cons}(\varphi)$, which is contradiction as $\mathfrak{N}^+, w \Vdash \bigwedge\vartheta_i$ by construction and $\mathfrak{N}^+, w \Vdash \neg\bigwedge\vartheta_i$. Just as above, this ensures the existence of some $\mathfrak{N}^* \in \mathrm{Mod}(\mathrm{Th}(\mathfrak{M}))$ and v such that $\mathfrak{N}^*, v \Vdash \Sigma \cup \{\varphi\}$.

We may assume that both $\mathfrak{N}^+$ and $\mathfrak{N}^*$ are generated by the points w and v respectively, since $\mathrm{Mod}(\mathrm{Th}(\mathfrak{M}))$ is closed under generated submodels, and also, they are $\aleph_1$-saturated. Now we can finish the proof more or less following the proof in ten Cate's work. We sketch the main argument. Define a binary relation Z between the elements of $\mathfrak{N}^+$ and $\mathfrak{N}^*$ as follows:

$$aZb \iff (\mathfrak{N}^+, a \Vdash \vartheta \iff \mathfrak{N}^*, b \Vdash \vartheta)$$

for all ϑ such that $\mathrm{Voc}(\vartheta) \subseteq \mathrm{Voc}(\varphi) \cap \mathrm{Voc}(\psi)$. By construction wZv, moreover, one can show that Z is a total bisimulation between $\mathfrak{N}^+$ and $\mathfrak{N}^*$. The zig-zag conditions are satisfied by $\aleph_1$-saturation, for the total bisimulation one uses the property that both sturctures are point generated. By $\mathfrak{F}$ and $\mathfrak{G}$ let us denote the underlying frames of $\mathfrak{N}^+$

and $\mathfrak{N}^*$ respectively. By Fact 4.2 and assumption there is a bisimulation product $\mathfrak{H} \in \mathcal{K}$. Since the projections $\pi_1: \mathfrak{H} \to \mathfrak{F}$ and $\pi_2: \mathfrak{H} \to \mathfrak{G}$ are bounded morphisms, we let $V(p) = \{u: \mathfrak{N}^+, \pi_1(u) \Vdash p\}$ for $p \in Voc(\psi)$ and $V(p) = \{u: \mathfrak{N}^*, \pi_2(u) \Vdash p\}$ for $p \in Voc(\varphi)$. Then $\langle \mathfrak{H}, V\rangle, \langle w, v\rangle \Vdash \varphi \wedge \neg\psi$, contrary to the assumption $\mathcal{K} \models \varphi \to \psi$.

□

We finish by adding the following remarks: although the conditions in Theorem 4.3 and Theorem 4.5 are quite independent, we could not find any classes where the logic generated by the class lacks Craig interpolation, but has the modelwise interpolation due to $\mathcal{K}$ being closed under modelwise bisimulation. In the future we would like to establish such results using the conditions in Theorem 4.5.

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